THE LINES AND PLANES CONNECTING THE POINTS OF A FINITE SET

BY

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0.1. Not later than 1933 I made the following conjecture, originally in the form of a statement on the minors of a matrix(1).

T_d. *Any n points in d-space that are not on one hyperplane determine at least n connecting hyperplanes.*

T_1 is trivial. It is easy to see (4.3) that T_d is a consequence of T_{d-1} and

U_d. *Any n points in d-space that are not on one hyperplane determine at least one ordinary hyperplane, that is, a connecting hyperplane on which all but one of the given points are on one linear* $(d-2)$*-space.*

In particular, T_2: n not collinear points are connected by at least n straight lines, is true if U_2 holds: for n not collinear points there is a straight line through only two of them. Now the nine inflexions of a plane cubic show that U_2 does not hold in the complex plane. Nevertheless H. Hanani gave in 1938 a combinatorial proof of T_2 for every (not only the real) projective plane. A greatly simplified version of this proof is given in 4.4. In 1939 A. Robinson proved U_2 for the real plane; in 1943 I found another very short proof (respectively the second and first proof in 1.1)(2). In 1944 I proved U_3 for real 3-space (1.4 and 1.5). U_d is still unproved.

The existence of three nonconcurrent ordinary lines is proved in 2.2, of four ordinary planes in 2.3. In §3 those n are determined for which U_2 holds for every field of coordinates F; 3.6 contains conditions for the fields F for which U_2 holds for every n. An application to configurations is made in §5.

Recently I learned that P. Erdös had dealt with U_2 (for the real plane) and T_2. U_2 had been conjectured by Sylvester (1893) and by Erdös (1933) and proved by T. Gallai (1933) and others (my proof being substantially that of R. Steinberg (1944)) (3). T_2 had been proved by G. Szekeres (1940 or 1941) and by N. G. de Bruijn and Erdös(4). A combination of part

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(1) *Beiträge zur Theorie der linearen Ungleichungen*, Dissertation, Basel, Jerusalem, 1936, p. 31.

(2) It is easy to give a finite turn to Robinson's proof: consider the smallest (in area) triangle with a line through vertex and base, and so forth.

(3) References in H. S. M. Coxeter, *The real projective plane*, New York, 1949, p. 26. Remark (Nov. 1950): L. M. Kelly's proof by a point and connecting line with minimal distance (it follows that the line is ordinary) holds also on the sphere and for the dual.

(4) *On a combinatorial problem*, Neder. Akad. Wetensch. vol. 10 (1948) pp. 421–423. Their question whether the number of ordinary lines tends to infinity with n is answered in the affirmative in 2.4.

of the latter proof with a combinatorial lemma (4.6) enabled me to prove T_d (4.5)[5].

0.2. We consider n different points P_i, $i=1, \cdots, n$, in the projective d-dimensional space or briefly d-space, and their *connecting lines* p_k, *connecting planes* π_m, and *connecting hyperplanes*, whose number is at most $C_{n,2}$, $C_{n,3}$ and $C_{n,d}$. As usual, connecting lines, meeting-points, and so on, are denoted by juxtaposition: PQ, pq, $p\cdot q$ (meeting point), $\pi\cdot q$, and so on.

1. Existence of an ordinary line or plane.

1.1. *If n points in the real projective plane are not on one straight line, then there exists a straight line containing exactly two of the points.*

Such a straight line is called an *ordinary line.*

Proof. Choose a straight line q that contains P_1 and no other P_i or $p_k\cdot p_l$. Let P_1Q, $Q=q\cdot p_1$, be a segment on q not met by any p_k, and P_2P_3 a segment on p_1 that contains Q and no P_i. Either p_1 is ordinary, or p_1 contains a point P_4. Then P_1P_4 is ordinary: for if it contained P_5 then P_2P_5 or P_3P_5 would meet the segment P_1Q.

Alternatively: proof of the dual theorem. Let P_1, P_2, P_3 be nonconcurrent straight lines. Either $P_2\cdot P_3$ is ordinary, or on a line P_4. Choose the line at infinity so that P_1, P_2, P_4 meet in finite points and that the point $P_1\cdot P_3$ is between $P_1\cdot P_2$ and $P_1\cdot P_4$. Either $P_1\cdot P_3$ is ordinary, or on a line P_5; P_5 divides either the triangle $\Delta P_1P_3P_2$ or $\Delta P_1P_3P_4$, say $\Delta P_1P_2P_3$. Either $P_2\cdot P_5$ is ordinary, or on a line P_6, which divides either $\Delta P_2P_5P_1$ or $\Delta P_2P_5P_3$. If this could be repeated ad inf., then the number of triangles, and of lines P_i, would be infinite.

The existence of an ordinary line would also follow from that of an *ordinary point,* that is, a point on at least $n/2$ connecting lines. The existence of such a point has not been proved or disproved.

1.2. *Invalid analogue in 3-space.* Consider the Desargues configuration of ten meeting-points of five planes $\alpha_1, \cdots, \alpha_5$ in general position. Among any three of these points there are necessarily two that have two planes in common, so that a plane through them must contain the third given point on their connecting line. Hence *there is no connecting plane containing exactly three of the points.* There exist planes through three points only, but these points are collinear.

Now consider $n_1\geqq 2$ on one and $n_2\geqq 2$ points on the other of two skew straight lines. Here every connecting plane contains at least min $(n_1, n_2)+1$ of the given points, and also every plane through more than two of the points contains at least min (n_1, n_2) points. Hence 1.1 cannot be extended to planes in 3-space by replacing "two" by "three" or "between 3 and k."

I know of no other examples with no connecting plane containing exactly three of the points.

[5] This and some other results were announced in an abstract of the present paper published in Bull. Amer. Math. Soc. Abstract 55-11-573.

1.3. *Valid analogues.*

(1) *If n points P_i in real projective 3-space are not on one plane, then there exists a connecting plane all of whose points P_i are on two straight lines through one of them, P_1, where P_1 may be chosen beforehand.*

This follows at once from 1.1. For project, from P_1, the other points P_i on a plane. In this plane there is a straight line l through exactly two of the projections. The plane P_1l fulfills the condition. (The projection must be made from a point P_i, for otherwise the plane obtained need not be a connecting plane.)

(2) *If n points in the real affine plane are not on one straight line or circle, then there is a straight line or circle containing exactly three of the points; one of the three points may, moreover, be chosen beforehand.*

For apply an inversion at the chosen point and 1.1.

There follows:

If n points on the real sphere are not on one plane, then there is a plane containing exactly three of them.

By a projective transformation the same holds for any quadric without straight lines. On ruled quadrics the proposition fails, by 1.2, since they contain pairs of skew lines.

The theorem is also an immediate consequence of (1) and holds for every convex surface that contains no straight segment.

1.4. In 1.5, we need the following lemma.

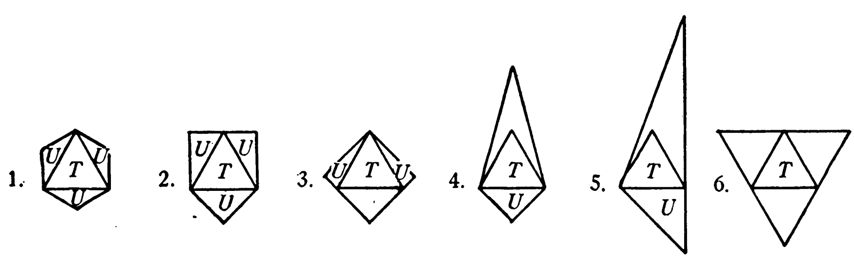
If $T = \triangle P_1P_2P_3$ is a closed triangle in the real projective plane, and if $P_4, \cdots, P_n$, $n > 3$, are points outside T, then at least one of the following statements is true:

(1) *Some connecting line p_k is outside T;*

(2) *Some p_k has exactly one point in common with T and contains exactly two points P_i;*

(3) *$n = 6$, and T is part of, and inscribed in, $\triangle P_4P_5P_6$.*

Proof. Make the plane affine by choosing a line at infinity outside T and through none of the points p_i. Let C be the smallest convex polygon containing all P_i. Any side of C that contains no vertex of T complies with (1). If every side of C contains a vertex of T, then C and T are situated as indicated in one of the following figures:



in which every triangular part denoted by U may also be missing.

Now in every triangle $U=\triangle P_1P_2P_4$ there is a side, say the segment P_1P_4, whose continuation does not belong to C. Then either P_1P_4 complies with (2), or the segment P_1P_4 contains a point P_5. Possibly $\triangle P_1P_2P_5$ contains a point P_6, and so on; at any rate we must, the number of points P_i being finite, at last arrive at a point P_m such that $\triangle P_1P_2P_m$ contains no P_i. If neither P_1P_m nor P_2P_m complies with (2), P_m will be on segments P_1P_j and P_2P_k, and P_jP_k complies with (1).

For case 6, $n>6$, proceed similarly.

In case 4 or 5 without U, with $C=\triangle P_1P_2P_4$, choose a new line at infinity near P_4 between P_4 and the segment P_1P_2, which reduces this case to one of the others, including case 6.

1.5. The true, nontrivial, analogue of 1.1 in 3-space is:

If n points P_i in real projective 3-space are not on one plane, then there exists a connecting plane such that all P_i but one on it are collinear.

Such a connecting plane is called an *ordinary plane.*

Proof. Choose a straight line q that contains P_1 and no other point of any $\pi_l\cdot\pi_m$. Let the segment P_1Q, $Q=q\cdot\pi_1$, be a segment on q not met by any π_m, and $T=\triangle P_2P_3P_4$ a triangle on π_1 that contains Q and no P_i. (Such a triangle is easily obtained: a triangle $\triangle P_jP_kP_l$ containing Q and some P_i is divided into triangles by segments connecting the P_i and the vertices, and the division is repeated as long as necessary.)

Now choose π_1 as the plane at infinity and P_1P_2, P_1P_3, P_1P_4 as axes for affine coordinates, oriented so that the coordinates of a point of the segment P_1Q are positive. Then every coordinate of every finite P_i is nonpositive; otherwise one of $P_iP_2P_3$, $P_iP_3P_4$, $P_iP_4P_2$ would meet the segment P_1Q.

Hence if π_1 contains a connecting line p_k outside T, then P_1p_k is a plane as required. The same is true if p_k has exactly one point in common with T and contains exactly two points P_i. If T is part of, and inscribed in, $\triangle P_5P_6P_7$, let $T_1=\triangle P_2P_5P_3$, $T_2=\triangle P_2P_5P_4$. For Q in T_1, take T_1 instead of T and $p_k=P_2P_4$; for T_2, take $p_k=P_2P_3$. Hence in every case of 1.4, 1.5 is proved; if π_1 contains no P_i, $i>4$, then π_1 is itself a plane as required.

There follows the existence of an ordinary line for n points in 3-space that are not on one straight line. But this follows already from 1.1, by projecting the n points from a point Q that is on no π_m. In the same way the existence of an ordinary line, or plane, is proved for d-space, $d>3$. The existence of an ordinary hyperplane remains still to be established.

2. Existence of more than one ordinary line or plane.

2.1. In every, not only the real, projective space of any number of dimensions we have immediately:

If every connecting line of n points P_i contains at least one of the points $Q_1, \cdots, Q_m$, and if the points $P_1, \cdots, P_r$ that are different from the points Q_j are not on one straight line, then $r\leqq(m-1)^2$.

Indeed, through either of P_1 and P_2 there are at most $m-1$ lines p_k besides P_1P_2; these lines meet in at most $(m-1)(m-2)$ points R_l other than Q_j. The points R_l include every point P_i, $i=3, \cdots, r$, that is not on P_1P_2, and if such a P_i exists then at most $m-1$ of $P_1, \cdots, P_r$ are on P_1P_2.

There is no obvious analogue for connecting planes.

If every connecting line of n points P_i contains one of the points Q_1, Q_2, then $n-1$ points P_i are on one straight line.

This is a special case of the preceding statement. Its analogue for planes is as follows.

If every connecting plane of n points P_i contains one of the points Q_1, Q_2, then either all P_i are on two straight lines or $n-1$ points P_i are on one plane.

For if $P_1, \cdots, P_4 \neq Q_1, Q_2$ are not on one plane, then we may suppose Q_1 on P_1P_2, Q_2 on P_3P_4. Any $P_5 \neq Q_1, Q_2$ must be on $P_1P_3P_2$ or $P_1P_3P_4$, and on $P_1P_4P_2$ or $P_1P_4P_3$, that is, on the quadrangle $P_1P_2P_3P_4$; likewise on $P_1P_2P_4P_3$; hence on P_1P_2 or P_3P_4.

2.2. A finite number of straight lines in the real projective plane that do not all pass through one point form a *proper division* of the plane into polygons.

If n points P_i in the real projective plane are not on one straight line and if P_1 is on no ordinary line, then the lines p_k that do not pass through P_1 form a proper division of the plane. In this division the sides of the polygon containing P_1 are ordinary lines.

For if the division were not proper, then, by 2.1, all P_i but one would be on one straight line, in which case every P_i is on an ordinary line. The second statement follows from the proof of 1.1.

If n points in the real projective plane are not on one straight line, then they determine at least three ordinary lines.

This follows from the preceding statement if there is a point P_1 that is on no ordinary line. But if every P_i is on one of two ordinary lines p_1 and p_2, and if $P_1 \neq p_1 \cdot p_2$ is on p_1 and $P_2 \neq p_1 \cdot p_2$ is on p_2, then P_1P_2 is also an ordinary line.

There follow at once corresponding refinements of the statements in 1.3.

If $P_2, \cdots, P_n$ in the real projective plane are not on one straight line, then there exists an ordinary line not through P_1.

Again this follows from the last statement but one if there is a point P_i that is on no ordinary line. But if every P_i is on an ordinary line, and if every ordinary line passes through P_1, then $P_2, \cdots, P_n$ determine no ordinary line and are therefore collinear.

If no $n-1$ of n given points in the real projective plane are collinear, then the ordinary lines determined by the n points are not concurrent.

For if they were, add their common point to the given points and apply the preceding statement.

By inversion we have:

If n points in the real affine plane are not on one straight line or circle, then there is a circle containing exactly three of the points; one of the three points may, moreover, be chosen beforehand.

2.3. *If n points in the real projective plane, or d-space, are not on one straight line and if they determine not more than m ordinary lines, then* $n \leqq C_{m+2,2}$.

For $d>2$, project on $(d-1)$-space from a point on no connecting plane; n and m remain unchanged.

For $d=2$, the ordinary lines divide the plane into at most $C_{m,2}+1$ polygons. In each polygon there is at most one point P_i (see proof of 2.2). Hence $n \leqq C_{m,2}+1+2m=C_{m+2,2}$.

Thus for sufficiently large n the number of ordinary lines exceeds any given number m.

2.4. *If n points in the real projective plane determine exactly three ordinary lines, then n equals* 3, 4, 6, *or* 7.

Proof. If the three ordinary lines are concurrent, then, by 2.2, $n-1$ points are collinear whence $n=4$.

If the ordinary lines form four triangles $T_1, \cdots, T_4$, and if T_1 contains a point P_i, then every connecting line that crosses T_1 passes through P_i. If T_1 contains P_1 and T_2 contains P_2, there can therefore be no P_i on the sides of T_1 and T_2 except the common vertices and points of P_1P_2. Hence the common vertices *are*, say, P_5 and P_6. The lines P_1P_5, and so on, are not ordinary, hence every T_i contains a point P_i; the last vertex is P_7. The points P_5, P_6, P_7 are the diagonal points of $P_1, \cdots, P_4$; $n=7$.

If T_1 contains P_1, and no other T_i a point P_i, then since every connecting line that crosses T_1 passes through P_1, and every connecting line through P_1 is not ordinary, we obtain easily that $n \leqq 7$. Hence the number m of connecting lines through P_1 is 2 or 3. If $m=2$, with $k \geqq 2$ and $l \geqq 2$ points besides P_1 on the connecting lines through P_1, then there are $kl \geqq 4$ ordinary lines. If $m=3$, with two points besides P_1 on each line, then there are easily seen to be two ordinary lines through one of those points; but if six points are on three ordinary lines, then each of them must be on only one ordinary line.

If no T_i contains a point P_i, then either $n=6$ (the meeting-points of four straight lines) or $n=3$.

2.5. *If n points in real projective d-space,* $d>2$, *are not on one hyperplane, then they determine at least* $C_{d+1,2}$ *ordinary lines; in fact more if* $n>d+1$.

Suppose this is true for $d-1$. Projecting from P_1 on a linear $(d-1)$-space we obtain points Q_i not on one linear $(d-2)$-space. Every line p_j projected into an ordinary line Q_1Q_2 is ordinary. If both, one, or neither of the projecting lines P_1Q_1 and P_1Q_2 is ordinary, then the number of these p_j is respectively one, at least two, or at least four. Therefore the total number m_P of ordinary lines p_j surpasses that of ordinary lines q_k at least by the number e of points Q_i situated on the ordinary q_k. Since there are at least $C_{d,2}$ and at most $C_{e,2}$ such q_k we have $e \geqq d$. Hence m_P can be exactly $C_{d,2}+d$ only if the number of *all* Q_i is d. But in this case every P_1Q_i that is not ordinary causes an increase in m_P, whence n must be $d+1$.

For $d=3$ the conclusions are obvious if every connecting line is ordinary. Otherwise take two connecting planes through a not ordinary line and a third connecting plane: they each carry at least three ordinary lines, and together, since only two meeting-lines can have been counted twice, at least seven ordinary lines.

That $m_P \geqq 6$ follows also from 2.7 (twelve lines less at most six meeting-lines, or one triple meeting-line and three meeting-lines, or one quadruple meeting-line).

If we try to prove the case $d=3$ by the same argument as $d>3$, we have, by 2.4, also to consider the possibility of seven points Q_i. But for P_2 and P_3 on P_1Q_4 and P_4 on P_1Q_5 either P_2P_4 or P_3P_4 would be ordinary.

Incidentally we see that this set of seven points is *rigid*, that is, it cannot be obtained from a set of points in 3-space that are not on one plane by projection from a point that is on no connecting plane; for such a projection leaves m_P unchanged. A set of points all but one of which are collinear is also rigid; so is any set for which every two p_k can be joined by a chain of adjacent triangles, that is, triangles with two common vertices and two common sides.

2.6. A finite number of planes in real projective 3-space that do not pass through one point form a *proper division* of space into polyhedra.

If no $n-1$ of n points P_i in real projective 3-space are on one plane and if P_1 is on no ordinary plane, then the planes π_m that do not pass through P_1 form a proper division of space. In this division the sides of the polyhedron containing P_1 are ordinary planes.

For if the division were not proper, then, by 2.1, all P_i would be on two straight lines, in which case every P_i is on an ordinary plane. The second statement follows from the proof of 1.5.

2.7. *If n points in real projective 3-space are not on one plane, then they determine at least four ordinary planes.*

Proof. By 2.6 we may suppose that every P_i is on an ordinary plane. (If $P_2, \cdots, P_n$ are on one plane and $n>4$ then there are at least four connecting planes through P_1.)

We now observe that if a point P_1 is on only one ordinary plane π_1, then there are, through P_1, three connecting lines that are not ordinary lines and not in one plane. (One of these lines may be in π_1.) For otherwise we would, by projecting from P_1 and applying 2.2, obtain another ordinary plane through P_1.

For n points with only two ordinary planes this leads to a contradiction. Suppose, therefore, that there are exactly three ordinary planes π_1, π_2, π_3, and that for $k=1, 2, 3$ all points P_i of π_k except P_k are on p_k. We then have to consider several cases.

(1) If a point P_i is on all three planes, then projection from P_i and application of 2.2 show easily that the above notations may be made in such a way that P_i is on p_1 and p_2.

(2) If P_i is on only π_1 and π_2, we see in the same manner that p_1 and p_2 pass through P_i and are not ordinary lines. Hence cases (1) and (2) exclude each other.

Now by our first remark there are, in case (1), three not coplanar, not ordinary connecting lines through P_1. This implies that P_i is also on p_3 and that P_1, P_2, P_3 are collinear. But then $P_iP_1P_2$ would be a fourth ordinary plane.

Applying the same remark, in case (2), to P_3 we see that P_3 is on the plane p_1p_2 and on the line P_1P_2. Now applying it to P_1 and to P_2 we find that p_3 is on p_1p_2, which is impossible.

Hence every P_i may be supposed to be on only one ordinary plane.

(3) If P_1, P_2, P_3 are collinear, application of the said remark to these points shows that P_1 is on p_2p_3, P_2 on p_3p_1, P_3 on p_1p_2. Hence p_1, p_2, p_3 are concurrent. But then every plane connecting P_4 on p_1 with points P_i on p_2 and p_3, except at most one plane, would be ordinary.

(4) Now application of the same remark to P_1 shows that P_1P_2 passes through a point P_3' on p_3; similarly $P_1' = P_2P_3 \cdot p_1$, $P_2' = P_3P_1 \cdot p_2$. If there were only two points P_i on p_3, then P_3' in place of P_3 in case (3) would lead to a contradiction. Applying the remark to the points P_i of p_3, we see that they are only P_3', P_3'' (on $p_1P_2P_3$), P_3''' (on $p_2P_1P_3$); similarly for p_1 and p_2. Since the line $P_1P_2P_3'$ is coplanar with P_3, P_1', P_2' and with P_3'', P_3''', P_2'', P_1''', and since P_3, P_1'', P_2''' are collinear, at least one of the planes $P_1P_2P_1''$ and $P_1P_2P_2'''$ would be ordinary.

2.8. Even more than in 2.4 it may be cumbersome to find the sets with exactly four ordinary planes, since it is not certain that n is bounded for these sets. Again things are simpler for more dimensions.

If n points in real projective 4-space are not on one hyperplane, then they determine at least ten ordinary planes; in fact more if $n > 5$.

If there is a not ordinary plane $P_1P_2P_3$, let $P_1, \cdots, P_5$ be not on one hyperplane. Then the four 3-spaces, other than $P_2P_3P_4P_5$, determined by these five points each carry at least four ordinary planes, and, since at most five meeting-planes are counted twice, together at least eleven ordinary planes.

If every connecting plane is ordinary, ten such planes are determined by any five points that are not on one hyperplane. A sixth point entails at least three more connecting planes.

3. Values of n for which an ordinary line must exist in every projective plane.

3.1. *For $n = 7$, and for every $n \geqq 9$, there is a Desarguesian projective plane* *and in it a set of n points P_i not all of which are on one straight line, such that every connecting line contains at least three points P_i.*

Proof. Consider the projective plane A_m over a finite field with m elements. Let B_m be the set of all the points on three straight lines through a point O of A_m, C_m $(m>2)$ the same set without O, D_m $(m>3)$ the set C_m plus two points not on a straight line through O, and E_m $(m>2)$ the set B_m plus an arbitrary set of points of which no straight line through O contains exactly one. Obviously $A_m, \cdots, E_m$ are sets as required, with respectively $n=m^2+m+1$, $n=3m+1$, $n=3m$, $n=3m+2$, $3m+3\leqq n\leqq m^2+m+1$.

Since for prime $m>6$ there is a prime p with $m<p<2m$, whence $3p+3<6m+3<m^2+m+1$, the values of n corresponding to E_m with prime m are 12, 13, and every $n\geqq 18$. We obtain $n=7$ by $A_2=B_2$, 9 by C_3, 10 by B_3, 12 by C_4 and E_3, 13 by B_4 and $A_4=E_3$, 14 by D_4, 15 by C_5 and E_4, 16 by B_5 and E_4, 17 by D_5 and E_4, 18 by E_5 and E_4.

For $n=11$, we try to find points $P_1, \cdots, P_5$ on one straight line, $P_1', \cdots, P_5'$ such that P_{k-1}', P_k, and P_{k+1}' (k mod 5) are collinear and that P_{k-2}', P_k, and P_{k+2}' are collinear, and P'' such that P_k, P_k', and P'' are collinear. Let $P''=(0,\ 0,\ 1)$, $P_k=(1,\ 0,\ a_k)$, $a_0=0$. We may suppose $P_0'=(1,\ 0,\ y)$, $P_1'=(1,\ a_1,\ a_1)$, $P_2'(1,\ a_2,\ a_2x)$, $P_3'=(1,\ a_3,\ a_3x)$, $P_4'=(1,\ a_4,\ a_4)$; then all conditions of collinearity with P'' or P_0 are fulfilled. From the collinearity of $P_1'P_2P_3'$ and of $P_2'P_3P_4'$ there follows easily that $a_2^{-1}-a_1^{-1}=a_4^{-1}-a_3^{-1}$, and from the collinearity of $P_0'P_1P_2'$ and $P_0'P_4P_3'$ there follows $a_1^{-1}-a_2^{-1}=a_4^{-1}-a_3^{-1}$; hence the characteristic of the field of coordinates must be 2. Then all conditions are fulfilled provided that $a_1^{-2}+a_2^{-2}+a_3^{-2}=a_1^{-1}a_2^{-1}+a_1^{-1}a_3^{-1}+a_2^{-1}a_3^{-1}$, $xa_3(a_1+a_2)=a_1(a_2+a_3)$, $y(a_1^{-1}+a_3^{-1})=1$; these equations have solutions such that $0, a_1, a_2, a_3, a_4$ are all different, though possibly only after an extension of the field.

3.2. *If n points in a projective plane are on two straight lines, but not on one straight line, then there are at least three ordinary lines.*

For if there are on either line at least two points P_i different from the meeting-point, then there are at least four ordinary lines. Otherwise $n-1$ points are on one straight line, and either $n>3$ with $n-1$ ordinary lines or $n=3$ with three ordinary lines.

If n points in a projective plane are not on a straight line, but $n-1$ of them are on two straight lines, then there are at least three ordinary lines.

For if there are on either line at least two points P_i different from the meeting-point (otherwise we come back to the preceding statement), then either three connecting lines are ordinary, or two connecting lines and some line through the nth point, or two connecting lines and the two given lines.

3.3. The above B_m $(m>1)$ and C_m $(m>2)$ may be constructed formally for any m (without imbedding in a projective plane) by considering points P_i, P_i', P_i'', $i=1, \cdots, m$, calling all P_i, all P_i', and all P_i'' collinear (in case B_m, with an additional point O), and defining m^2 collinear triples $P_iP_j'P_k''$, for example, by the condition $i+j+k\equiv 0 \pmod{m}$.

If n points in a projective plane are on three straight lines, but not on one straight line, and if they do not form a set B or C, then there are at least three ordinary lines.

Proof. By 3.2 we may assume that each line contains two points P_i that are on neither of the other two lines. Then, if two of the given lines do not contain the same number of points P_i, there is, through every point P_i on the third line, and not common to all three lines, at least one ordinary line. Hence there certainly exist three ordinary lines unless either all three lines contain the same number of points P_i, or they contain 2, 2, and $j>2$ points P_i besides a common meeting-point. In the latter case there are at least $4j-8\geqq 4$ ordinary lines; in the former, if there is a point P_i common to only two of the lines, or if there is a point P_i common to all three lines and if the given points are not a set B, and finally if every P_i belongs to only one of the lines and if the given points are not a set C, we easily establish the existence of at least three ordinary lines.

3.4. *If n points in a projective plane are not on one straight line, but if there is a straight line q containing at least $n-5$ of the points, then either there are at least three ordinary lines, or $n=7$, the points being four points and their three diagonal points and the field of coordinates having the characteristic* 2, *where there is no ordinary line* (case B_2).

For if the points P_i are on three straight lines, apply the preceding proposition. Otherwise there are five points not on q, and no three of them are collinear. Through every point P_i on q there is an ordinary line, and for $n\leqq 7$, at least six of the ten lines connecting the five points are ordinary.

There follows:

If n points in a projective plane are not on one straight line ($n=3$, 4, 5, 6, 8), *then there are at least three ordinary lines.*

Seven points in a projective plane determine either at least three ordinary lines, or none; the latter happens only for seven collinear points and for points forming a set B_2.

We have also:

Nine points in a projective plane determine either at least three ordinary lines, or none; the latter happens only for nine collinear points and for points forming a set C_3.

For we may suppose that no straight line contains more than three of the given points. If there is an ordinary line P_1P_2, then through P_1 and P_2 there is a second and third ordinary line. If there is no ordinary line, let P_1, P_2, P_3 be collinear. It is easily seen that P_4 is collinear with two points P_5 and P_6, and P_7 with P_8 and P_9. Thus we can apply 3.3.

3.5. Note that there exist sets with exactly one or two ordinary lines. For example A_m ($m>2$) less all points of one straight line but two is a set with one ordinary line; here n is at least 11. Similarly all points of A_m ($m>3$), not on P_1P_2 and P_1P_3, together with P_1, P_2, P_3, form a set with two ordinary lines; n is at least 15.

3.6. Let F be the field of coordinates. We may ask for conditions on F such that n points not on one straight line shall necessarily determine an ordinary line. So far one sufficient and two necessary conditions are known.

The sufficient condition for F is that it be a (possibly non-Archimedean) ordered field. In this case the same proofs hold as for the field of real numbers.

One necessary condition is that the characteristic of F be 0. The configuration A_p (p prime) occurs for characteristic p (and only for it). Likewise $p \neq 0$ implies, for every $d > 1$, the existence of a finite point set in d-space without ordinary hyperplanes (defined in 0.1).

The other necessary condition is that F contain no roots of unity other than ± 1. (This implies again that p is 0, 2, or 3.) For let $\{a\}$, $\{b\}$, $\{c\}$ be finite subsets of F without 0 such that every product ab is in $\{c\}$, every bc in $\{a\}$, every ca in $\{b\}$; a simple computation shows that one of them, say $\{a\}$, consists of the m mth roots of 1 for some m, and either $\{b\} = \{c\} = \{a\}$ or $\{b\} = \{c\} = -\{a\}$. For $m > 2$ the $3m$ points $(0, -1, a)$, $(b, 0, -1)$, $(-1, c, 0)$ form a set of the C_m type and determine no ordinary line.

4. **Existence of at least n connecting lines, planes, or hyperplanes.**

4.1. *If n points in the real projective plane are not on one straight line, then they determine at least n connecting lines.*

Proof. For $n \leqq 2$ there is nothing to prove. For $n > 2$ suppose the theorem true for $n-1$. Among the n given points consider one which is on an ordinary line. The other $n-1$ points are either on one straight line or determine at least $n-1$ connecting lines. In both cases the contention follows immediately.

By a remark at the end of 1.5 the theorem holds equally for points in real projective d-space, $d > 2$.

Similarly one has:

The number of connecting lines is n only if $n-1$ of the points are on one straight line. It can be $n+1$ only for $n=5$ and $n=6$, $n+2$ only for $n=4$, $n=6$, and $n=7$.

4.2. *If n points in real projective 3-space are not on one plane, then they determine at least n connecting planes.*

Proof. For $n \leqq 3$, there is nothing to prove. For $n > 3$, suppose the theorem true for $n-1$. Among the n given points, consider one which is on an ordinary plane such that the other given points on that plane are collinear. The other $n-1$ points are either on one plane or determine at least $n-1$ connecting planes. In the latter case the contention follows immediately; in the former apply 4.1.

Again the theorem holds for points in real projective d-space, $d > 3$.

Similarly:

The number of connecting planes is n only if the points are on two straight *lines. It can be* $n+1$ *only if* $n-1$ *points are on one plane and if, moreover,* $n=6$ *or* $n=7$.

4.3. Using the abbreviations T_d and U_d of 0.1 we have:

If U_d *and* T_{d-1} *hold then* T_d *holds, and the number of hyperplanes connecting n points in projective d-space is exactly n if and only if the points are, for odd d, on* $(d+1)/2$ *straight lines, and for even d, all but one on* $d/2$ *straight lines.*

The proof of the first statement is as for 4.2. (It is true anyway, because of 4.5.) To prove the second part for $d=2c-1$ note that for c straight lines $p_1, \cdots, p_c$, with $n_i>1$ points $P_{i1}, \cdots, P_{in_i}$ on p_i such that all these points are not on one hyperplane, every point P_{ik} together with the lines p_j, $j\neq i$, determines a hyperplane π_{ik}; hence the number of connecting hyperplanes is the same as that of the given points. Now if a new point P_l is neither on π_{11} nor on π_{21}, then $P_l, P_{11}, P_{21}, p_3, \cdots, p_d$ determine a new hyperplane, different from any other similarly defined new hyperplane. Hence the number of new hyperplanes can be exactly one only if either P_l is on every π_{ik}, $i\neq i_0$, whence P_l is on p_{i_0}, or P_l is on every π_{ik} except say π_{11} and π_{21}, whence $n_1=n_2=2$, P_l on $P_{12}P_{22}$; in this case take $P_{11}P_{21}=p_1'$, $P_{12}P_{22}=p_2'$. The proof for $d=2c$, $c>1$, is exactly "on the same lines," with an additional point P_0 and hyperplane $\pi_0=(p_1\cdots p_c)$.

4.4. *If n points* P_i *in a projective plane are not on one straight line, then they determine at least n connecting lines.*

For suppose there are m connecting lines $p_1, \cdots, p_m$, $1<m\leqq n$, with n_k+1 given points on p_k, $n-2\geqq n_1\geqq n_2\geqq \cdots \geqq n_m\geqq 1$. Besides p_1 there are at least $(n-n_1-1)/n_2$ connecting lines through every P_i on p_1, whence $m\geqq(n-n_1-1)(n_1+1)/n_2+1$; by $n\geqq m$ we have

$$n_1(n_1+1)\geqq(n-1)(n_1+1-n_2). \tag{1}$$

Hence if $n_2=1$, then $n_1+1\geqq n-1$; $n-1$ points P_i are on one straight line and $m=n$. For $n_2>1$ let P_1 be a given point on p_1 and if possible also on p_2. The other points P_i on p_1 and p_2 are connected by at least n_1n_2 straight lines. Taking into account, as before, the connecting lines through P_1, we have $n\geqq(n-n_1-1)/n_2+n_1n_2+1$, that is, $n(n_2-1)\geqq n_1n_2^2+n_2-n_1-1$ or $n\geqq n_1(n_2+1)+1$. By (1), $n_1+1\geqq(n_2+1)(n_1+1-n_2)$ or $n_2\geqq n_1$, whence $n_1=n_2$, $n\geqq n_1^2+n_1+1$. Again by (1), $n\leqq n_1^2+n_1+1$, whence

$$n=n_1^2+n_1+1. \tag{2}$$

Now let $n_1=\cdots=n_f>n_{f+1}$. Of the $C_{n,2}$ pairs of points P_i, there are $C_{n,2}-C_{n_1+1,2}f$ pairs that determine a straight line different from $p_1, \cdots, p_f$; hence the number of these lines is at least $(C_{n,2}-C_{n_1+1,2}f)/C_{n_1,2}$, whence $n\geqq m\geqq f+(C_{n,2}-C_{n_1+1,2}f)/C_{n_1,2}$. There follows $(n-f)n_1(n_1-1)\geqq n(n-1)-fn_1(n_1+1)$ or, by (2), $2fn_1\geqq 2nn_1$, $n\leqq f$. By $f\leqq m\leqq n$ we have $m=n$ as desired.

Since $f=m$, there are n_1+1 points P_i on every p_k. The meeting-point of any two p_k is a P_i, since otherwise $m>(n_1+1)^2=m+n_1$. The n given points form, therefore, in this case themselves a whole projective plane, and if this plane is Desarguesian, then m_1 is a power of the characteristic of the field of coordinates.

Another proof is as for the following generalization.

4.5. T_d. *If n points in a projective d-space are not on one hyperplane, then they determine at least n connecting hyperplanes.*

T_1 is trivial. By T_d-1 we have $\bar{h}\leqq\bar{p}$, where $\bar{h}$ is the number of given points on one of the connecting hyperplanes h and $\bar{p}$ the number of connecting hyperplanes through one of the given points p that is not on h. Now apply the lemma 4.6.

It follows by projection on $(m+1)$-space, possibly after extension of the field of coordinates, that n points in a projective d-space that are not on one m-dimensional linear manifold determine at least n such manifolds. In particular, n points in real spherical d-space that are not on one m-dimensional sphere determine at least n such spheres.

4.6. L_e. *If n not necessarily different nonvoid subsets of a set of e elements have the property that for every given subset p and element h not in p we have $\bar{h}\leqq\bar{p}$ ($\bar{h}$ being the number of given subsets containing h and $\bar{p}$ the number of elements of p) and if no element belongs to all the subsets, then $e\geqq n$.*

L_1 is void, L_2 trivial. If $n>e>2$ we show that there exists a proper subsystem $P=(p_1, \cdots, p_e)$ of the given system of subsets such that h_i is not in p_i, $i=1, \cdots, e$. Then we would have $\bar{h}_i\leqq\bar{p}_i$, $\sum\bar{h}_i\leqq\sum\bar{p}_i$, while $\sum\bar{h}_i$ equals the sum of *all* $\bar{p}$. In order to see that P ought to exist, we make use of the well known theorem (Frobenius and others) that an n-row determinant of zeros and indeterminates is zero if and only if the matrix contains a rectangle of $j\cdot k$ zeros, $j+k=n+1$. We apply this to a matrix $(a_{\mu\nu})$, μ, $\nu=1, \cdots, n$, where $a_{\mu\nu}=0$ if and only if $\mu\leqq e$ and h_μ is in p_ν and see that P exists except if k sets p contain the same j elements $h_1, \cdots, h_j$, $j+k=n+1$. Now for $j=1$, h_1 would belong to all the given sets. For $1<j<e$ we have $j\leqq n-2$, $k\geqq 3$. Omit $k-2$ from among the above k sets, and in each of the remaining $n-k+2$ sets delete all elements other than $h_1, \cdots, h_j$. The system obtained fulfills the conditions of L_j: if one of its subsets were void all would be void, since $k-2\geqq e-j$ if $e\leqq n-1$; and $\bar{h}\leqq\bar{p}$ implies $\bar{h}-(k-2)\leqq\bar{p}-(e-j)$. Hence $j\geqq n-(k-2)$ which is not so. Finally, if $j=e$, we omit one of the k sets, delete one of the elements, and apply L_{e-1}.

5. **Remark on complete configurations.** A configuration (a_b, c_d) of a points and c straight lines in a projective plane, any point of which is on $b>1$ lines and on any line of which there are $d>1$ points (whence $ab=cd$), is *line-complete* if it contains every straight line through two of its points; the dual configuration is *point-complete*. Since among $C_{a,2}$ pairs of points, $cC_{d,2}$ are connected by lines of the configuration, the condition for line-complete-

ness is $c = c_{a,2}/C_{d,2}$. By 1.1 we have:

In the real plane the only line-complete configuration is the trivial configuration a_{a-1}, $(C_{a,2})_2$. *Only for* $a=3$ *is this configuration also point-complete.*

This contains a simple, purely geometrical proof of the nonrealizability of $c_3 = (9_4, 12_3)$ (the inflexions of a cubic) in the real plane.

To find the combinatorial formula of all line-complete configurations note that $b=(a-1)/(d-1)$ whence $a=1+b(d-1)$, $c=(1+b(d-1))b/d$; putting $e=b^2-c=b(b-1)/d$, the formula becomes $1+b(d-1)_b$, b^2-e_d. The first cases with $d>2$, ordered by ascending $ab=cd$, are

$$1\colon 7_3,\ 7_3;\quad 2\colon 9_4,\ 12_3;\quad 3\colon 16_3,\ 8_6;\quad 4\colon 13_4,\ 13_4;\quad 5\colon 13_6,\ 26_3;$$

$$6\colon 16_5,\ 20_4;\quad 7\colon 21_4,\ 14_6;\quad 8\colon 15_7,\ 35_3;\quad 9\colon 21_5,\ 21_5;\quad 10\colon 25_6,\ 30_5.$$

Of these, 3 and 7 are not realizable in any projective plane, because of $c<a$ and 4.4.

If the configuration shall also be point-complete then $b^2-b(b-1)/d=1+d(b-1)$; subtracting with b and d interchanged we have $(b-d)(b+d)(b-1)\cdot(d-1)=0$, whence $b=d$. We obtain the configurations $A_{b-1}=(b^2-b+1_b, b^2-b+1_b)$ of 3.1 and 4.4, including the above cases 1, 4, 9. From such "finite projective planes," finite affine planes such as 2, 6, 10 are obtained by deleting one line and its points. Case 8 can be realized by projecting a finite projective 3-space.

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